Excercises

1. $\int_0^\infty \frac{(\sin x)}{x^\alpha} dx$ exists for $0 < \alpha < 2$.

Hint – Integration by parts and use comparison test.

2. Show that $\int_0^1 x^p \sin(\frac{1}{x}) dx$ exists for $p > -2$.

Hint – put $x = \frac{1}{y}$ apply example 1.

3. Show that $\int_1^\infty x^\alpha \, dx$ does not exist for any real $\alpha \geq -1$.

Hint : Use $\lim_{b\to\infty} \int_1^b x^\alpha \, dx$

4. Show that $\int_0^1 \frac{\sin x}{x^\alpha} dx$ exists for all $\alpha < 2$.

Hint : Use $0 \leq \sin x \leq x$ on $[0, \frac{\pi}{2}]$ and compare with $\int_0^1 x^{1-\alpha} \, dx$ which exists for $1-a > -1$

5. Find the value of the Improper Integral $\int_0^1 \left(1-t^2\right)^{-\frac{1}{2}} dt$

Hint : For 0< a < 1 , $\int_0^a \left(1-t^2\right)^{-\frac{1}{2}} dt = \int_0^{\sin^{-1} a} dx = \sin^{-1} a$, *putting* $t = \sin x$. Now let $a \to 1-$

6. Show that for $4b > a^2$, $\int_{-\infty}^{\infty} \frac{dx}{x^2 + ax + b} = \frac{\pi}{\sqrt{b - \frac{a^2}{4}}}$

Hint : Let $4b - a^2 = t^2$, $t > 0$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+ax+b} = 4\int_{-\infty}^{\infty} \frac{dx}{y^2+t^2} = \frac{\pi}{\sqrt{b^2-\frac{a^2}{4}}}$$

7. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2(n^2+1)}$ is convergent with sum 1.

Hint : $\frac{1}{n^2\left(n^2+1\right)} = \frac{1}{n^2} - \frac{1}{n^2+1}$ & $\sum \frac{1}{n^2\left(n^2+1\right)} = \sum_{1}^{\infty} \frac{1}{n^2} - \sum_{1}^{\infty} \frac{1}{n^2+1}$

8. Show that the series $\sum_{n=1}^{\infty} \frac{1}{\left(\sqrt{n+\alpha}+\sqrt{n+1+\alpha}\right)}$ is divergent when $\alpha > 0$.

Hint : $\frac{1}{\sqrt{n+\alpha}+\sqrt{n+1+\alpha}} = \sqrt{n+1} - \sqrt{n}$ *, then*

$$\sum_{1}^{m} \frac{1}{\sqrt{n+\alpha}+\sqrt{n+1+\alpha}} = \sqrt{m+1} - 1 \to \infty \text{ as } m \to \infty$$

9. Show that the series $\frac{1}{5}+\frac{1}{10}+\frac{1}{15}+\frac{1}{30}+\frac{1}{45}+\frac{1}{90}+.....$ is convergent.

Hint : $a_1 = \frac{1}{5}$ *then* $a_n \begin{cases} \frac{1}{2}a_{n-1} \text{ if } n \text{ is even} \\ \frac{2}{3}a_{n-1} \text{ if } n \text{ is odd} \end{cases}$

$$\frac{a_{n+1}}{a_n} \le \frac{2}{3} \text{ .therefore } \sum_{1}^{\infty} a_n \text{ converges}$$

10. Show that $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\alpha} r^n$ is convergent for any real α if $0 < r < 1$.

Hint : $\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^{\alpha} r \to r \text{ as } n \to \infty$

11. Show that the series $\sum_{2}^{\infty} \frac{2}{\left(n\log n^2\right)^{\alpha}}$ is convergent for $\alpha > 1$, divergent for $\alpha \le 1$.

Hint : Consider $f(x) = \dfrac{1}{x(\ln x^2)^{\alpha}}$ on $[2, \infty]$ put $y = \log x^2$, then

$$\int_2^b \frac{dx}{x(\log x^2)^{\alpha}} = \frac{1}{2}\int_{\ln 4}^{\ln b^2} \frac{dy}{y^{\alpha}},\ \alpha > 1 \text{ converges}$$

12. Show that the series $1 - 1 + \frac{1}{2} - \frac{1}{2^3} + \frac{1}{3} - \frac{1}{3^3} + \ldots..$ is not convergent.

Hint : sum of 2m terms in the form $\sum_1^m \frac{1}{n} - \sum_1^m \frac{1}{n^3}$

13. Show that the function

$$f(x) = \begin{cases} x\sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x^2}, \text{when } x \neq 0 \\ 0, \text{when } x = 0 \end{cases}$$

is not R-integrable in any interval containing the origin.

Hint: The function $f(x)$ is unbounded in a small neighbourhood of $x = 0$

14. Show that the function $f(x) = 0$ when $x \neq 0$ & $f(0) = 1$ is R-integrable.

Hint: $f(x)$ is continuous everywhere except $x = 0$ and the value of integral is 0.

15. Define $\log x = \int_1^x \frac{dt}{t}$, prove that

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to 0} x^{\alpha} \log x = 0, \text{ where } \alpha > 0.$$

Hint: Since $\frac{1}{t} < \frac{1}{t^{1-a}}$ when $t > 0$ & $a > 0$

$$\therefore \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{t^{1-a}} \text{ or } \log x < \frac{x^a - 1}{a} < \frac{x^a}{a}$$

$$\therefore \frac{\log x}{x^{\alpha}} < \frac{x^{a-\alpha}}{a} \to 0 \text{ as } x \to \infty$$

for a can be chosen to be less than α.

II. put $t = \frac{1}{u}$

16. if $f(x), \phi(x)$ be both R-integrable and such that $|f(x)| \le |\phi(x)|$ for every value of x, then prove that

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |\phi(x)| dx$$

17. Show that if $Q(x)$ decreases to zero as $x \to \infty$, then the improper Integrals

$\int_1^\infty Q(x)\sin x \cdot dx$ & $\int_1^\infty Q(x)\cos x \cdot dx$ are convergent.

18. Let $f(x) = \begin{cases} x & when\ is\ \ rational \\ 1-x & when\ is\ irrational \end{cases}$

Show that $f(x)$ assumes every value between 0 and 1 once and only once as x increases from 0 to 1, but is discontinuous for every value of x except $x = \frac{1}{2}$.

19. Find the nature of discontinuity of the following functions at $x = 0$.

(i) $f(x) = [-x^2]$ where $[x]$ denotes the greatest integer not greater than x

(ii) $f(x) = \begin{cases} \frac{|x|}{x} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$

Hint: (i) $f(+0) = -1 = f(-0)$, $f(0) = 0$

Therefore $f(x)$ has removable discontinuity at $x = 0$.

(ii) $f(+0) = 1$, $f(-0) = -1$, $f(0) = 0$

Therefore $f(x)$ has discontinuity of the 1st kind , at $x = 0$.

20. Examine the function defined by

$f(x) = x^2 \cos\left(e^{\frac{1}{x^2}}\right)$, $x \ne 0$, $f(0) = 0$ with regard to (i) continuity (ii) differentiability and (iii) continuity of the derivatives, in the interval (-1,1).

Hint : (i) continous as $\lim_{x\to 0} f(x) = 0 = f(0)$ and also at all other points .

(ii) $f'(x)$ is exist at $x = 0$.

$f'(x) = 2x\cos\left(e^{\frac{1}{x}}\right) + e^{\frac{1}{x}}\sin(e^{\frac{1}{x}}),\ x \neq 0$

As $x \to 0+,\ f'(x)$ does oscillates

$x \to 0-,\ f'(x) \to 0$, So

$f(x)$ is differentiable everywhere in the interval (-1,1) but has a discontinuity of the 2^{nd} kind on the right at $x = 0$.

21. If $f''(x) > 0$ for all values of x, prove that

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$$

Hint : Use Taylor's expansion upto order 2 by choosing for $\frac{x_1 + x_2}{2}$ & h for $\frac{x_1 - x_2}{2}$